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Optimal Inspection Schedules for Failure Detection When Tests Hasten Failures

by
N. Wattanapanom
L. Shaw

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lambda(i) > lambda(i-1)

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1. Introduction

Barlow, Hunter and Proschan [1, 2, 3] posed and solved an interesting class of inspection problems related to reliability and maintainability of systems. In their models, a system operates for random time until it fails, but the failure can be detected only after a test costing c_1 has been performed. For example, judgment of whether or not a radar system's detection and false-alarm probabilities are acceptable might require connection of special test equipment and injection of special testing signals.

One approach to this problem seeks optimal testing times to minimize a loss function of the form

$$L = E\left\{c_1 N + c_2 d\right\} \tag{1.1}$$

in which N is the number of tests until the first one after the failure, and d is the time between the failure time and the subsequent detection time t_N. Reference 3 gives algorithms for finding the optimal inspection times t_i for lifetime distributions which are uniform, exponential or Polya of order 2, when using the single-cycle (no renewal) loss function in (1. 1).

A second point of view postulates repair or replacement after each failure detection, with associated average cost and average repair time requirements. In this case the performance is judged using the mean cost per unit time or equivalently, [14] the ratio of mean-cost-per-renewal to mean-time-between-renewals. Reference 3 shows how techniques used for the single-cycle problem can be modified to find the best testing times for the case with renewals.

Here we seek similar solutions but include the possibility that the mechanical and electrical stresses of the test might destory or damage the system. Mathematically, we allow the ith test to destroy the system with

probability β or, with probability $(1-\beta)$, to increase the failure rate (reciprocal of mean time-to-failure) to $\lambda_i > \lambda_{i-1}$ without changing the form of the conditional lifetime distribution. These generalizations are approached in a direct manner by using dynamic programming to set up recursive optimization equations. The main contribution here is the presentation of convergent algorithms for solving the optimization equations.

Section 2 examines exponential conditional lifetime distributions, and Section 3 gives similar results for uniform conditional distributions. Section 4 discusses extensions to more general distributions, etc.

2. Exponential Conditional Lifetime Distributions

We consider a single component system with two conditions <u>good</u> or <u>failed</u>. The system's status can be determined only by a test costing c_1 units a failed system introduces a penalty of c_2 per unit time during the interval between the failure time t and the subsequent inspection time t_N. The inspection times (to be optimized) are denoted by $t_0, t_1, \ldots$ with interchecking times defined as $\delta_\kappa = (t_{\kappa+1} - t_\kappa)$.

The exponential conditional lifetime assumption enters through the family of conditional failure time densities

$$\text{(i)}\quad \bar{f}_\kappa(t \mid t_\kappa < t) = \lambda_\kappa e^{-\lambda_\kappa (t-t_\kappa)} \quad ; \; t > t_\kappa \tag{2.1}$$

$$\text{(ii)}\quad f_\kappa(t \mid t_\kappa < t \leq t_{\kappa+1}) = \frac{\lambda_\kappa e^{-\lambda_\kappa (t-t_\kappa)}}{1-e^{-\lambda_\kappa \delta_\kappa}}$$

While any sequence of test-dependent failure rate parameters might be used, the algorithms to be used below require that $\lambda_\kappa \to \infty$ as the number κ of possible tests increases without bound.

2.1 Single-Cycle Exponential Case

We consider first the case in which a test never causes immediate failure ($\beta=0$) and in which performance is judged over a single failure cycle. When including test induced performance degradation it is reasonable to add another cost function reward term $(-c_3)$ per unit time of good operation. Otherwise an optimal policy might call for several quick tests which increase the probability of short lifetime but also decrease the mean time between failure and detection.

With N as the random variable denoting the index of the test which actually detects a failure, the loss to be minimized by the choice of testing times can be expressed as

$$L_0(t_1, t_2 \ldots) = E\left[c_1 N + c_2(t_N - t) - c_3(t-t_0) \mid t>t_0\right] \qquad (2.2)$$

Since the testing times after t_κ can influence only future contribution to the mean cost, it is useful to consider the optimal mean future loss L_κ^o at any testing time t_κ. This can be expressed recursively as

$$L_\kappa^o = \min_{t_{\kappa+1}} \left\{ (1-e^{\lambda_\kappa \delta_\kappa}) \int_{t_\kappa}^{t_{\kappa+1}} \left[c_1 + c_2 (t_{\kappa+1} - t) - c_3(t-t_\kappa)\right] f_\kappa(t)\, dt \right.$$

$$\left. + e^{-\lambda_\kappa \delta_\kappa} \left[c_1 + c_2 (t_{\kappa+1} - t) + L_{\kappa+1}^o\right] \right\} \qquad (2.3)$$

in terms of the two cases $t_\kappa < t < t_{\kappa+1}$ or $t > t_{\kappa+1}$.

The expression for L_κ^o in (2.3) simiplifies to

$$L_\kappa^o = \min_{\delta_\kappa} \left\{ c_1 - (c_2+c_3)/\lambda_\kappa + c_2 \delta_\kappa + e^{-\lambda_\kappa \delta_\kappa} (L_{\kappa+1}^o + (c_2+c_3/\lambda_\kappa)) \right\} \qquad (2.4)$$

A recursive equation for the optimal interchecking time δ^o_κ follows from simple differentiation.

$$\delta^o_\kappa = \frac{1}{\lambda_\kappa} \ell n \left[\frac{\lambda_\kappa}{c_2} L^o_{\kappa+1} + 1 + c_3/c_2 \right] \qquad (2.5)$$

Substitution of (2.5) into (2.4) yeilds the companion relation

$$L^o_\kappa = c_1 - c_3/\lambda_\kappa + c_2 \delta^o_\kappa \qquad (2.6)$$

At first glance (2.5) may not appear to be useful necessary condition, since it is defined in terms of the next optimal loss, which is as yet unknown. Futhermore, the structure of the problem does not permit definition of a "last" step from which we can work backwards. However, the assumed growth of λ_κ does permit indirect evaluation of the "last step" optimal cost as c_1, as follows. As κ and λ_κ grow very large, $P[\, t < t_\kappa + \Delta \mid t > t_\kappa] \rightarrow 1$ for any $\Delta > 0$. Thus, in that limit, $t^o_{\kappa+1} \rightarrow t_\kappa$ and $L^o_\kappa \rightarrow c_1$.

More formally,

<u>Lemma 1.</u> $\lim_{\kappa \to \infty} L^o_\kappa = c_1$

<u>Proof:</u> The future mean loss for any (not necessarily optimal) sequence of testing times may be written similarly to (2.4) as

$$L_\kappa(\delta_\kappa, \ldots) = c_1 - (c_2+c_3)/\lambda_\kappa + c_2 \delta_\kappa + e^{-\lambda_\kappa \delta_\kappa}(L_{\kappa+1} + (c_2+c_3)/\lambda_\kappa) \qquad (2.7)$$

The limiting expression

$$\lim_{\kappa \to \infty} L_\kappa = c_1 + c_2 \delta_\kappa$$

is clearly minimized by $\delta_\kappa = 0$, with the corresponding limiting loss stated in the lemma.

The foregoing boundary condition on L^o_κ suggests the following algorithm for computation of the optimal testing intervals in relation to a given failure rate sequence $\lambda_0 < \lambda_1 < \lambda_2 \ldots$

1. Choose an initial maximum number of tests, M_1.
2. Assume $L^o_\kappa(M_1) = c_1$ and use (2.5) and (2.6) to compute

$$\delta^o_{M_1-1}(M_1),\ L^o_{M_1-1}(M_1),\ \delta^o_{M_1-2}(M_1), \ldots.\ \delta^o_0(M_1),\ L^o_0(M_1)$$

3. Repeat for $M_2 > M_1$, etc.
4. Stop when $\delta^o_i(M_{\kappa+1})$ and $\delta^o_i(M_\kappa)$ are sufficiently close in respective values for $1 \leq i \leq I$, and a suitable I. (I is specified with respect to a sufficiently high value for $P[\ t < t_I.\ ]$)

With regard to sufficiency of the zero-derivative condition underlying (2.5), it is easy to show that $\delta_\kappa > 0$ implies

$$L^o_{\kappa+1} > - c_3/\lambda_\kappa \tag{2.8}$$

and that (2.9) insures that the second derivative of the bracketed term in (2.4) is positive.

<u>Examples</u>

1. The first example has $c_1 = 1$, $c_2 = 20$, $c_3 = 20$, $\lambda_0 = 2$ and

$$\lambda_\kappa = \lambda_0 / (0.9)^\kappa$$

i.e each test reduces the mean residual lifetime by a factor of 0.9. Table 1 shows the approximate values for δ^o_κ and L^o_κ which result from $M_1 = 21$. Similar calculations with $M_2 = 31$, $M_3 = 41$ etc produced the same 10 digits for the first seven δ_i and L_i values. This degree of convergence will be adequate if the resulting $P[\ t < t_7] = 1 - \exp[\ -\sum_0^6 \delta_i \lambda_i] = 0.98+$ is acceptable.

Table 1 also shows how the tests reduce the mean lifetime from 0.5 (in the absence of testing) to 0.438. The $E_\kappa(T)$ shown there is defined as the mean lifetime given exactly κ tests, using the δ_κ values found above. This assumes, for the purpose of approximate caluclations, that a failure after time t_κ goes undetected. The convergent E_κ sequence can be computed recursively from the relation [5]

$$E_\kappa(T) = E_{\kappa-1}(T) - \left(\frac{1}{\lambda_{\kappa-1}} - \frac{1}{\lambda_\kappa} \right) e^{-\sum_{0}^{\kappa-1} \lambda_i \delta_i}$$

Similar calculations with other parameter values reveal interesting trends. Interchecking times are smaller if λ_0 is increased, if c_2 is increased, or if c_3 is decreased. If c_3 is large compared to c_1 and c_2 then there is a premium on long life and L^o_κ is a monotonically increasing function of initial failure rate. This is demonstrated in Figure 1 where it should be noted that, e.g., the loss after t_1, when having started with λ_0 at t_0, is equivalent to the loss after t_0 when starting with (λ_0/ρ). $(L_1(\lambda_0) = L_0(\lambda_0/\rho))$ However, Figure 2 shows that if c_3 is relatively smaller, then increasing the initial failure rate may increase or decrease the mean loss.

Table 2 shows results analogous to those in Table 1 except that here the failure rate grows linearly instead of geometrically:

$$\lambda_\kappa = \lambda_0 \, (1+\kappa)$$

2.2 Optimum Inspection with Renewals- Exponential Case

The foregoing model will now be generalized to allow for repairs or renewals which each have a mean cost s and a mean completion time of r time units. Inspections times between renewals must again be selected, but performance is now measured in terms of the average cost per unit

time. The i^{th} cycle, starting at time t_{0_i} and ending at t_{N_i}, will accrue a cost

$$C_i = N_i c_1 + d_i c_2 + s \tag{2.9}$$

and will have a duration

$$\tau_i = (t_{N_i} - t_{0_i}) + r \tag{2.10}$$

Using these terms the average cost is defined as

$$R(\delta) \triangleq \lim_{K \to \infty} \frac{\sum_{i=1}^{k} C_i}{\sum_{i=1}^{k} \tau_i} \tag{2.11}$$

The argument δ is shown to emphasize the dependence of this average cost on an inspections policy $\delta = \{ \delta_0, \delta_1 \ldots \}$.

It is well known [4] that $R(\delta)$ can be expressed in terms of the mean cost per cycle $\overline{C}$ and the mean cycle duration $\bar{\tau}$, as

$$R(\delta) = \bar{C} / \bar{\tau} \tag{2.12}$$

Both the numerator and demonenator of (2.12) are affected by the inspection policy δ. However, it is possible to find the δ which minimizes $R(\delta)$ by considering an auxiliary optimization problem having the cost function

$$\mathcal{L}(\mu, \delta) = \bar{C} - \mu \bar{\tau} \tag{2.13}$$

These two problems are related as follows.

<u>Theorem 1:</u>[3] If there exists a $\mu = \mu^*$ for which

$\min_{\delta} \mathcal{L}(\mu^*, \delta) \triangleq \mathcal{L}(\mu^*, \delta(\mu^*)) = 0$, then the schedule $\delta(\mu^*)$ also minimizes $R(\delta)$.

The existance of such a desired μ^* is quite evident for the present problem. Equation (2.13) can be rewritten as

$$L(\mu, \delta) = c_1 \bar{N} + c_2 \bar{d} + s - \mu [\bar{\tau} + r] \qquad (2.14)$$

a little thought shows that when $\mu \geq c_2$, increases in δ_i will decrease $\mathcal{L}$ since $\bar{\tau} > \bar{d}$. Thus, for $\mu \geq c_2$ the optimal policy is to make no inspections or repairs, with a corresponding optimal $\mathcal{L} < 0$. On the other hand, when $\mu = 0$, $\mathcal{L} > 0$ for any policy - - including the optimal one. We conclude that $\mathcal{L}(\mu, \delta,(\mu))$ changes from a positive value at $\mu = 0$ to a negative one for $\mu = c_2$, and this change must be a continuous one passing through zero at least once in view of the smooth nature of all densities and cost terms entering into the loss functions.

An algorithm which uses Theorem 1 to optimize inspection time in the presence of renewals consists of two parts. An algorithm like the one in Section 2.1 is needed to minimize (2.14) for each choice of μ until $\mathcal{L}(\mu, \delta(\mu)) = 0$. The first part parallels the approach in the single-cycle case discussed earlier since $\mathcal{L}$ defined in (2.14) is a single cycle loss with μ similar to c_3.

In particular (2.14) leads to the following recursive expression for the minimal mean future loss after test κ,[5]

$$L^o_\kappa = \min_{\delta_\kappa} \Big\{ c_1 + (c_2 - \mu)\,\delta_\kappa - c_2/\lambda_\kappa + (s - \mu r)$$

$$+ e^{-\lambda_\kappa \delta_\kappa} (L^o_{\kappa+1} - s + \mu r + c_2/\lambda_\kappa) \Big\} \qquad (2.15)$$

Differentiation yields the recursive optimization equations.

$$\text{(i)} \quad \delta_\kappa^o = \lambda_\kappa^{-1} \ln \left[\left(c_2 + \lambda_\kappa L_{\kappa+1}^o - \lambda_\kappa (s - \mu r) \right) / (c_2 - \mu) \right] \tag{2.16}$$

$$\text{(ii)} \quad \mathcal{L}_\kappa^o = c_1 - \mu/\lambda_\kappa + (s - \mu r) + (c_2 - \mu)\, \delta_\kappa^o$$

It can be shown [5] that the $\mu < c_2$ condition derived above insures that the zero-derivative condition corresponds to a minimum and that all $\delta_\kappa^o > 0$. Finally, Lemma 1 must be modified for the present loss function to

<u>Lemma 2:</u> $\lim\limits_{\kappa \to \infty} \mathcal{L}_\kappa^o = c_1 - \mu r + s$ (2.17)

Once the μ^* of Theorem 1 has been found, it follows immediately that

$$\min_\delta R(\delta) = \mu^* \tag{2.18}$$

<u>Sensitivity</u>

The preceding paragraphs describe how to get the best inspection schedule δ and the corresponding minimal loss per unit time. It is also of interest to see how sensitive this solution procedure is to the precision of knowledge of model parameters, e.g. λ_0, and how sensitive it is to the accuracy of the iterative calculations. It is straight forward to use the exponential conditional densities to get interative expressions for $\bar{N}, \bar{d}$ and $\bar{\tau}$ which can be terminated when the additional terms contribute very little. (See for example, the E_κ expression in the example of Section 2.1)[5]

<u>Examples</u> Tables 3, 4 and 5 show results for numerical examples based on the models in this section. Table 3 shows the small sensitivity of $R(\delta)$ to use of non-optimal testing schedules that result when $\mu \neq \mu^*$. The parameters for Tables 4 and 5 differ only in that the latter has non-zero values for r and s, the renewal costs. Both of those tables show minimum R values for several λ_0 as well as the performance degradation when $\lambda_0 = 5$ but, when the optimum δ-schedule for a different λ_0 value is used.

3. Uniform Conditional Distributions

In this section we consider a failure model in which a single-component system begins operation at t_0 with a uniform failure time distribution between t_0 and $(t_0 + T_0)$. An inspection test at time $t_1 = (t_0 + \delta_1)$ of a "good" system leaves a remaining failure distribution uniform between t_1 and $t_1 + T_1$ where $T_1 = \rho(T_0 - \delta_1)$, $0 < \rho < 1$. That is, after each test the conditional distribution is still uniform, but with its mean lifetime reduced by a factor ρ, as compared to the case of no test.

The analysis of this uniform conditional distribution model is not so neat as that of exponential models, but convergent algorithms have been found for determining the optimum number and locations of inspection tests for given to and ρ when the single-cycle loss function of (2.2) is to be minimized.

Dynamic programming is used to optimize the location of the first of n tests $(t_1(n) = t_0 + \delta_1(n))$ assuming that all future tests are optimally located. Difficulties arise when a zero-derivative condition yields an inadmissible $\delta_1(n) > T_0$ or when the remaining lifetime T_1 is so short that the analytical minimal future loss expression $L^o_{n-1}(T_1)$ usedfor computing $\delta_1(n)$ is no longer valid. These pathologies and techniques for circumventing them are best described by the following small example.

The best location for a simgle test is found by minimizing the following mean loss expression with respect to x_1 ($x_1^o = \delta_1$). (We introduce the extra symbol x to help distinguish between zero-derivative intervals and optimal intervals).

$$L_1(T_0, x_1) = c_1 + \frac{x_1}{T_0} \int_0^{x_1} \frac{1}{x_1} [c_1(x_1 - t) - c_3 t] \, dt$$

$$+ (1 - \frac{x_1}{T_0}) [- c_3 x_1 + \rho \frac{(T_0 - x_1)}{2} (c_2 - c_3)] \quad (3.1)$$

This reduces to

$$L_1(T_0, x_1) = x_1^2 (c_2 + c_3 - a_0)/2 T_0 - x_1(c_3 - a_0) + c_1 - a_0 T_0/2 \qquad (3.2)$$

in which we use

$$a_0 = \rho (c_3 - c_2) \qquad (3.3)$$

The minimum of the parabolic function (3.2) is clearly achieved by $x_1(1)$

$$x_1(1) = \frac{c_3 - a_0}{c_2 + c_3 - a_0} T_0 = \delta_1(1) \qquad (3.4)$$

which lies between 0 and T_0 since $c_3 - a_0 > 0$. The corresponding minimum loss is

$$L_1^o (T_0) = c_1 - \frac{a_0 T_0}{2} - \frac{T_0(c_3 - a_0)^2}{2(c_2 + c_3 - a_0)} \qquad (3.5)$$

In order to compute the best two-test policy we use the foregoing one-test solution for the last step, and choose the first test interval as the x_1 which minimizes

$$L_2^o (T_0) = c_1 + \min_{x_i} \Bigg\{ \frac{x_1}{T_0} \int_0^{x_1} \frac{1}{x_1} [c_2(x_1 - t) - c_3 t] \, dt$$

$$+ (1 - \frac{x_1}{T_0}) [- c_3 x_1 + L_1^o (\rho (T_0 - x_1))] \Bigg\} \qquad (3.6)$$

The bracketed expression in (3.6) is a parabolic function of x_1 with a minimum at

$$x_1(2) = \frac{c_1 + (c_3 - a_1)}{c_2 + c_3 - a_1} T_0 \qquad (3.7)$$

where

$$a_1 = \rho \left[a_0 + (c_3 - a_0)^2/(c_2 + c_3 - a_0) \right]$$

However, if $T_0 < c_1/c_2$ then $x_1(2) > T_0$ is not an admissible testing interval. In such a case, the shape of the x_1 - function indicates that $\delta_1(2) = T_0$. It follows that

$$\delta_2(2) = T_0$$

and

$$L_2^o(T_0) = 2c_1 + L_0^o(T_0) \tag{3.8}$$

so the best use of two inspections costs more than using no inspections at all, when $T_0 < c_1/c_2$.

We have just seen how optimization of the first of 2 tests might appear to yield an infeasible solution. Building on this, a related difficulty arises when comparing different locations for $x_1(3)$ in such cases. The remaining lifetime for the best two tests may become so small that it is better to replace them with no tests.

Careful consideration of all such details led to an algorithm based on interative calculation of the sequence of zero-derivative intervals $x_1(n)$, and a related sequence of number S_n. These sequences are computed recursively for $n = 1, 2, \ldots$ according to formulas given in the appendix, until

$$x_1(n) > T_0 - S_n \tag{3.9}$$

If n^* is the smallest n to satisfy inequality (3.9) then the optimal number of tests is found by searching for the minimum of the previously computed mean losses for each number of tests less than n^*. After thus determining the optimal number of tests, their locations can be computed from formulas given

in the appendix.

The adequacy of this algorithm is verified by a set of lemmas and theorems which establish:[5]

$$
\begin{aligned}
&\text{(i)} \quad x_1(n) > 0 \\
&\text{(ii)} \quad x_1(n) \geq T_0 - S_n \text{ implies } L_n^o \geq L_{n-2}^o \qquad (3.10) \\
&\text{(iii)} \quad x_1(n) > T_0 - S_n \text{ implies } x_1(m) > T_0 - S_m \\
&\qquad\qquad \text{for all } m > n
\end{aligned}
$$

Example

Figure 3 summarizes the resjuts of one example in which the optimal number of tests in 4 ($n^* = 7$) and the testing reduces the mean lifetime from 10 to 47.31.

4. Summary

Tests which degrade performance have been introduced into a standard model of inspection time optimization. Convergent algorithms have been developed for single-cycle and mean loss rate cases when the conditional lifetime distributions are exponential. The uniform conditional distribution model was solved for a single-cycle case. No results have been found concerning the rates of convergence of these algorithms, but all examples have produced rapid convergence, as evidenced by those included here.

Several extensions of these results are straightforward. An algorithm for the mean loss rate version of the uniform distribution case follows directly from Theorem 1 and the single-cycle results. Inclusion of a non-zero probability β_κ that the κ^{th} inspection causes immediate destruction is easy. For

example the only change thus introduces into (2.6) is replacement of $L^o_{\kappa+1}$ by $(1 - \beta_{\kappa+1}) L^o_{\kappa+1}$.

The general approach to the exponential case, with backward recursion from an approximate "last-step" loss is applicable to other conditional lifetime distribution of infinite support. Other distribution might require search-type optimizations at each step, rather than the simple zero-derivative condition, but no other special properties of the exponential distribution were used in the analysis. This backward recursion might also be efficient for getting approximate solutions to problems without degrading tests, by imposing a small mean lifetime reduction factor in order to have a "last-step" loss.

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7. Appendix

This appendix lists formulas used to carry out the algorithm outlined in Section 3, all summations have values of zero when their lower limits exceed their upper limits.

$$a_0 = \rho(c_3 - c_2); \; a_n = \rho\left[a_{n-1} + (c_3 - a_{n-1})^2/(c_2 + c_3 - a_{n-1})\right]; \; n = 1, 2\ldots$$

$$b_0 = 0; \; b_n = (c_3 - a_n)\left(n\,c_1 - \sum_{i=1}^{n-1} b_i\right)/(c_2 + c_3 - a_n); \; n = 1, 2\ldots$$

$$d_0 = 0; \; d_n = \left(n\,c_1 - \sum_{i=1}^{n-1} b_i\right)^2 / 2\rho\,(c_2 + c_3 - a_n); \; n = 1, 2\ldots$$

$$S_1 = S_2 = 0$$

$$S_n = [\ (n-2)\,c_1 - \sum_{i=1}^{n-3} b_i + (c_2 + c_3 - a_{n-2})\,S_{n-1}] / \rho\, c_2$$

$$x_1(n) = [\ (n-1)\,c_1 - \sum_{i=1}^{n-2} b_i + (c_3 - a_{n-1})\,T_0] \,/ (c_2 + c_3 - a_{n-1})$$

$$x_i(n) = [\ (n-i)\,c_1 - \sum_{j=1}^{n-i-1} b_j + (c_3 - a_{n-i})\,T_{i-1}] \,/ (c_2 + c_3 - a_{n-i})$$

$$i = 1, 2, \ldots n$$

8. References

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4. S. Ross, Applied Probability Models with Optimization Applications, Holden Day, 1970

5. N. Wattanapanom, "Optimal Inspection Schedules When Tests Hasten Failure," Dissertation for Ph. D. in Operations Research, Polytechnic Institute of New York, June 1976.

$c_1 = 1,\ c_2 = 20,\ c_3 = 20,\ \lambda_0 = 2,\ \rho = 0.9,\ M = 21$

κ	λ_κ	δ^o_κ	L^o_κ
20	16.4505462646	0.0630758378	1.0457516582
19	14.8054933548	0.0689164386	1.0274788134
18	13.3249425888	0.0741102386	0.9812602189
17	11.9924449921	0.0793027668	0.9183387025
16	10.7932004929	0.0847316170	0.8416138568
15	9.7138824463	0.0905008817	0.7511086328
14	8.7424926758	0.0966715266	0.6457534650
13	7.8632432175	0.1032920655	0.5239778407
12	7.0814170837	0.1104095565	0.3838976640
11	6.3732748032	0.1180737114	0.2233700960
10	5.7359466553	0.1263393089	0.0400034017
9	5.1623516083	0.1352671649	-0.1688600729
8	4.6461162567	0.1449250417	-0.4061697551
7	4.1815042496	0.1553888160	-0.6751914360
6	3.7633543015	0.1667435399	-0.9795371479
5	3.3870182037	0.1790842460	-1.3232139609
4	3.0483160019	0.1925172210	-1.7106551576
3	2.7434844971	0.2071614750	-2.1467697772
2	2.4691352844	0.2231501702	-2.6369982959
1	2.2222213745	0.2406318986	-3.1873654603
0	2.0000000000	0.2597726583	-3.8045468330

			$E_\kappa(T)$
0	2.0000000000	0.2597726583	0.5000000000
1	2.2222213745	0.2406318986	0.4702605553
2	2.4691352844	0.2231501702	0.4545806209
3	2.7434844971	0.2071614750	0.4464467698
4	3.0483160019	0.1925172210	0.4423000139
5	3.3870182037	0.1790842460	0.4402246966
6	3.7633543015	0.1667435399	0.4392063376
7	4.1815042496	0.1553888160	0.4387169919
8	4.6461162567	0.1449250417	0.4384870213
9	5.1623516083	0.1352671649	0.4383814642
10	5.7359466553	0.1263393089	0.4383342074
11	6.3732748032	0.1180737114	0.4383136019
12	7.0814170837	0.1104095565	0.4383048639
13	7.8632432175	0.1032920655	0.4383012655
14	8.7424926758	0.0966715266	0.4382998288
15	9.7138824463	0.0905008817	0.4382992734
16	10.7932004929	0.0847316170	0.4382990659
17	11.9924449921	0.0793027668	0.4382989911
18	13.3249425888	0.0741102386	0.4382989651
19	14.8054933548	0.0689164386	0.4382989564
20	16.4505462646	0.0630758378	0.4382989535

Table 1: Single-cycle Exponential Case

Geometric Failure Rate Increase

$c_1 = 1,\ c_2 = 20,\ \lambda_0 = 2,\ \lambda_\kappa = \lambda_0\,(1+\kappa),\ M = 21$

κ	λ_κ	δ^o_κ	L^o_κ
20	42.0000000000	0.0335949262	1.1957080478
19	40.0000000000	0.0369912863	1.2398257256
18	38.0000000000	0.0387230923	1.2481460571
17	36.0000000000	0.0401703517	1.2478514777
16	34.0000000000	0.0416523429	1.2448115629
15	32.0000000000	0.0432567596	1.2401351929
14	30.0000000000	0.0450239817	1.2338129679
13	28.0000000000	0.0469890663	1.2254956109
12	26.0000000000	0.0491933456	1.2146361424
11	24.0000000000	0.0516901414	1.2004694939
10	22.0000000000	0.0545509078	1.1819272475
9	20.0000000000	0.0578743458	1.1574869156
8	18.0000000000	0.0618015925	1.1249207391
7	16.0000000000	0.0665429831	1.0808596611
6	14.0000000000	0.0724284308	1.0199971880
5	12.0000000000	0.0800096095	0.9335255225
4	10.0000000000	0.0902906597	0.8058131933
3	8.0000000000	0.1053210944	0.6064218879
2	6.0000000000	0.1300346553	0.2673597733
1	4.0000000000	0.1798829585	-0.4023408294
0	2.0000000000	0.3364124000	-2.2717519999

κ	λ_κ	δ^o_κ	$E_\kappa(T)$
0	2.0000000000	0.3364124000	0.5000000000
1	4.0000000000	0.1798829585	0.3724337071
2	6.0000000000	0.1300346553	0.3517262875
3	8.0000000000	0.1053210944	0.3469810713
4	10.0000000000	0.0902906597	0.3457550888
5	12.0000000000	0.0800096095	0.3454237559
6	14.0000000000	0.0724284308	0.3453331474
7	16.0000000000	0.0665429831	0.3453084953
8	18.0000000000	0.0618015925	0.3453018835
9	20.0000000000	0.0578743458	0.3453001446
10	22.0000000000	0.0545509078	0.3452996974
11	24.0000000000	0.0516901414	0.3452995852
12	26.0000000000	0.0491933456	0.3452995577
13	28.0000000000	0.0469890663	0.3452995512
14	30.0000000000	0.0450239817	0.3452995497
15	32.0000000000	0.0432567596	0.3452995493
16	34.0000000000	0.0416523429	0.3452995492
17	36.0000000000	0.0401703517	0.3452995492
18	38.0000000000	0.0387230923	0.3452995492
19	40.0000000000	0.0369912863	0.3452995492
20	42.0000000000	0.0335949262	0.3452995492

<u>**Table 2: Single-cycle, Exponential Case**</u>

<u>Linear Failure Rate Increase</u>

$c_1 = 1.0$, $c_2 = 20.0$, $s = 0$, $r = 0$, $\lambda_0 = 5.0$, $M = 21$, $\rho = 0.9$, $\kappa = 20$, and

$$\lambda_\kappa = \lambda_0 / \rho^\kappa$$

	μ	R
	10	12.72591
	11	12.67058
	12	12.63822
$\mu^* =$	12.63183	12.63200
	13	12.63429
	14	12.66670
	15	12.74756
	16	12.89658
	17	13.14914
	18	13.57903
	19	14.39155

Table 3: Mean Loss Rate Minimization - Exponential Case

$c_1 = 1.0$, $c_2 = 20$, $s = 0$, $r = 0$, $\rho = 0.9$, $M = 21$, and $\lambda_\kappa = \lambda_0/\rho^\kappa$

λ_0	$R(5)_{mismatch}$	R_{min}
2	13.05727	8.68520
3	12.75710	10.27669
4	12.65410	11.55473
5	12.63183	12.63200
6	12.64438	13.56575
7	12.66983	14.38974
8	12.69736	15.12581
9	12.72094	15.78896
10	12.73731	16.38998

Table 4: Minimum Loss Rates - Exponential Case; Sensitivity to λ_0 error

$c_1 = 1.0$, $c_2 = 20.0$, $s = 1.2$, $r = .001$, $\rho = 0.9$, $M = 21$ and $\lambda_\kappa = \lambda_0/\rho^\kappa$

λ_0	$R^{(5)}_{mismatch}$	R_{min}
2	16.28924	10.62738
3	16.22188	12.87934
4	16.21368	14.70045
5	16.21360	16.21360
6	16.24733	17.47788
7	16.38830	18.52204
8	16.38830	19.35293

Table 5: Mean Loss Rates - Exponential Case

Sensitivity to λ_0 Error - Non Zero Renewal Costs

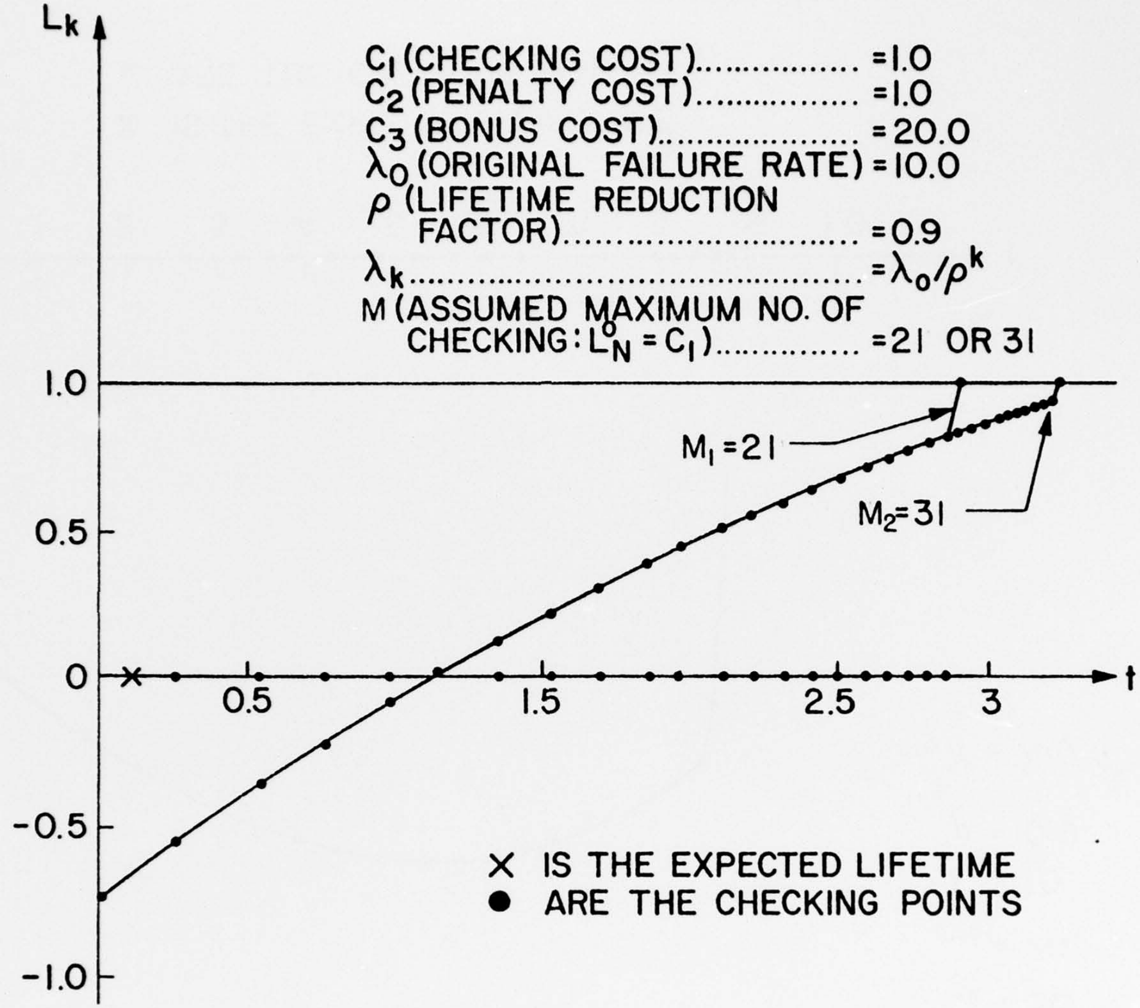


Fig. 1 - Single Cycle Exponential Case

L_k^o increase with λ_k

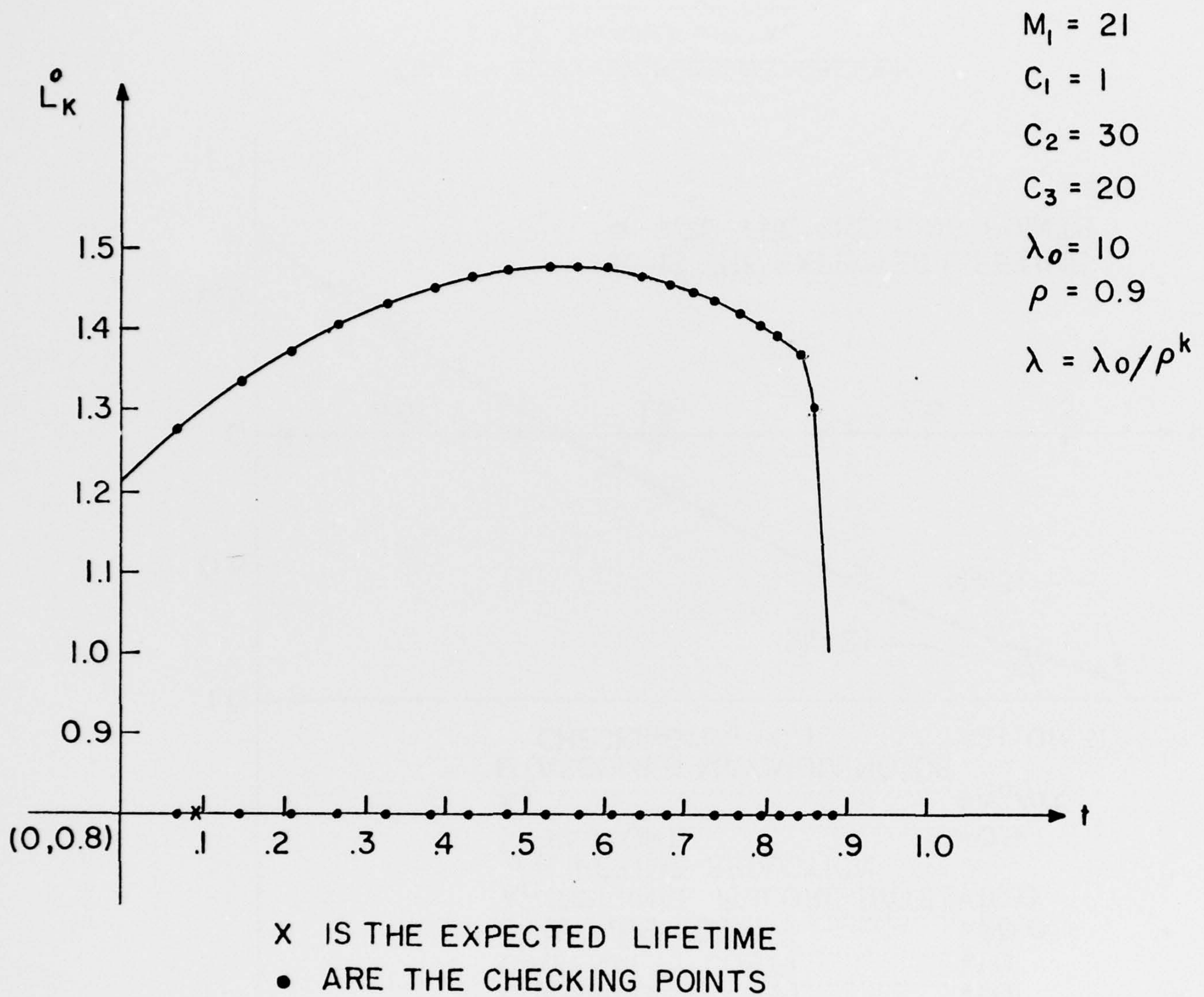
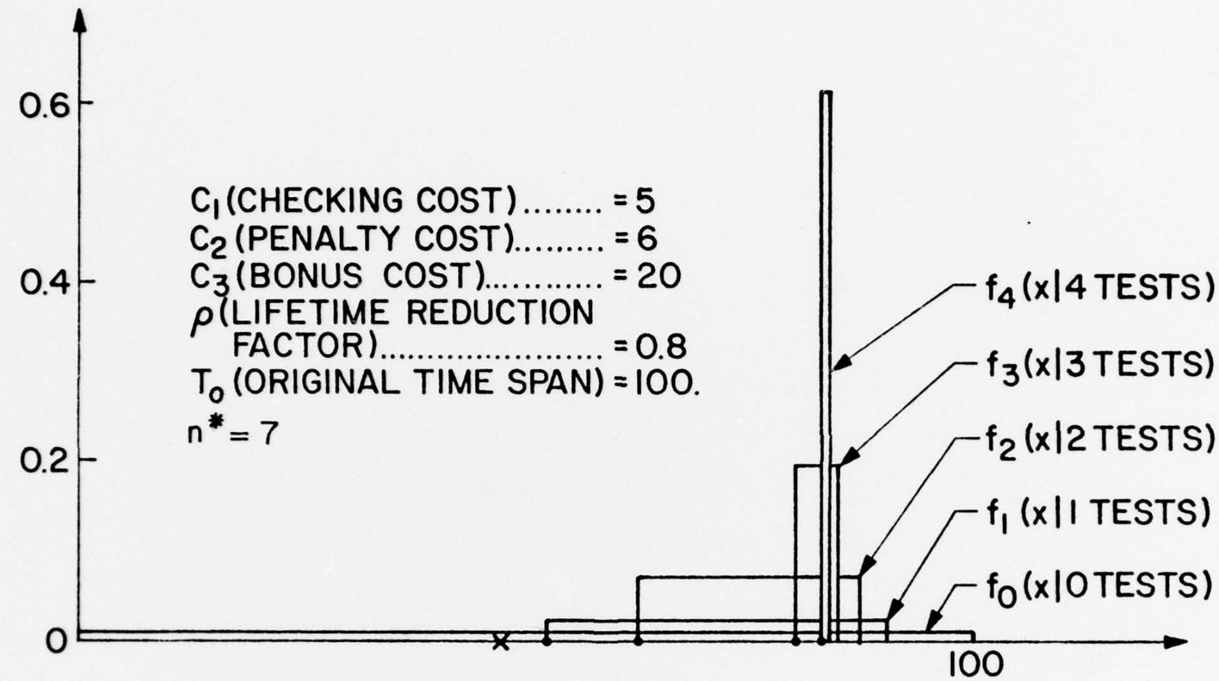


Fig. 2. - Single Cycle Exponential Case
L_k^o dependence on λ_k



× IS THE EXPECTED LIFETIME 47.31

● ARE THE CHECKING POINTS 52.71, 73.05, 80.90 AND 83.82

Fig. 3 - Uniform Conditional Lifetime Distributions and Optimal Testing Times